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Three-dimensional contact problems for an elastic wedge with a coating☆

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ABSTRACT

Three-dimensional contact problems for an elastic wedge, one face of which is reinforced with a Winkler-type coating with different boundary conditions on the other face of the wedge, are investigated. A power-law dependence of the normal displacement of the coating on the pressure is assumed. The contact area, the pressure in this region, and the relation between the force and the indentation of a punch are determined using the method of non-linear boundary integral equations and the method of successive approximations. The results of calculations are analysed for different values of the aperture angle of the wedge, the relative distance of the punch from the edge of the wedge, the ratio of the radii of curvature of the punch (an elliptic paraboloid), and the non-linearity factors of the coating. The results obtained are compared with the solutions of similar problems for a wedge without a coating.

Problems of the action of a specified load on one face of a three-dimensional wedge for different boundary conditions on its other face, and also a number of contact problems for a three-dimensional wedge without a coating were investigated in Refs 1, 2. The method of non-linear boundary integral equations[3,4] is an effective method of solving three-dimensional contact problems with an unknown contact area. This method has been used for a wedge both ignoring friction,[5] and taking into account friction forces perpendicular to the edge of the wedge.[6] It is well known that a non-linear coating can simulate a rough surface for contact,[7] in which case the contact area can be specified.[8] Three-dimensional contact problems with an unknown contact area for a rough elastic layer taking friction forces into account have been considered[9] using the method of successive approximations. When the punch is remote from the edge of the wedge the contact-pressure distribution under the punch begins to be determined by the solution of the contact problem for an elastic half-space with a coating, which has been investigated previously in several publications (see, for example, Ref. 10, Section 3.5).

1. Formulation of the problem and its reduction to the solution of a non-linear integral equation

We will consider, in cylindrical coordinates r, φ, z, a three-dimensional elastic wedge $\{0 \le r < \infty, 0 \le \varphi \le \alpha, |z| < \infty\}$ with elastic characteristics G (the shear modulus) and ν (Poisson's ratio). The z axis coincides with the edge of the wedge. The face of the wedge $\varphi = 0$ is under sliding conditions or is rigidly clamped or stress-free (Problems A, B and C respectively). Suppose that, by the action of a force P, applied at a distance H from the edge of the wedge, a rigid punch is pressed into the face $\varphi = \alpha$. The punch has the form of an elliptic paraboloid, the base of which is described bythe function

$$f(r, z) = (r - a)^2/(2R_1) + z^2/(2R_2)$$

We will assume, for simplicity, that the problems are symmetrical about the z coordinate and $R_1 \le R_2$. As a result of the action of the force P the punch penetrates by an amount δ and rotates by an angle γ around the edge of the wedge. We will neglect the friction forces in the contact area. There is a non-linear Winkler-type coating between the punch and the face of the wedge (Ref. 7, p. 363), which may simulate the roughness of the contact surface. We will use a power law of the relation between the contact pressure $\sigma_\varphi = -q(r, z)$ and the normal displacement of the coating. The deformation of the coating makes an additional contribution u_φ^a to the normal displacement in the contact area Ω, defined by the formula

$$u_\varphi^a = -Aq^\beta(r, z), \quad (r, z) \in \Omega, \quad 0 < \beta < 1 \tag{1.1}$$

The quantity A represents the pliability and thickness of the coating (the coarseness of the processing of the rough surface).

The contact Problem A for the wedge with sliding clamping is equivalent to the symmetrical contact problem for two punches on different faces of a wedge of double the aperture angle.

For specified quantities α, G, ν, δ, γ, A, β and known function f(r,z), it is required to obtain the contact area Ω, the pressure q(r,z) in this region, and also the quantities P and H.

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The contact condition has the form

$$u_\varphi^a + u_\varphi = -[\delta + \gamma r - f(r,z)], \quad (r,z) \in \Omega \tag{1.2}$$

For a displacement u_φ of the face of the wedge due to the action of the pressure, on the basis of the results for a smooth wedge[1] the following formulae hold

$$u_\varphi(r,\alpha,z) = -\frac{1}{2\pi\theta}\int_\Omega K(r,z,x,y)q(x,y)d\Omega, \quad \theta = G(1-\nu)^{-1}$$

$$K(r,z,x,y) = R^{-1} + K_*(r,z,x,y),$$

$$R = [(r-x)^2 + (z-y)^2]^{1/2} \tag{1.3}$$

For Problems A (m = 1) and B (m = 2)

$$K_*(r,z,x,y) = \frac{4}{\pi^2}\int_0^\infty\int_0^\infty \operatorname{sh}\pi u\left\{K_{iu}(sx)[W_m(u,\alpha) - \operatorname{cth}\pi u] + \frac{W_m(u,\alpha)}{\operatorname{ch}(\pi u/2)}\Psi_m(u,sx)\right\}K_{iu}(sr)\cos(z-y)sdsdu \tag{1.4}$$

The functions $\psi_m(u, sx)$ ($m = 1, 2$) are found from Fredholm integral equations of the second kind ($0 \le u < \infty$ for a fixed value of sx)

$$\Psi_m(u,sx) = (1-2\nu)\int_0^\infty L(u,\tau)\left[\Psi_m(\tau,sx) + \operatorname{ch}\frac{\pi\tau}{2}K_{i\tau}(sx)\right]d\tau \tag{1.5}$$

where

$$L(u,\tau) = 2\operatorname{ch}\frac{\pi u}{2}\operatorname{sh}\frac{\pi\tau}{2}W_m(\tau,\alpha)\int_0^\infty \frac{\operatorname{sh}\pi t\, g_m(t,\alpha)dt}{(\operatorname{ch}\pi t + \operatorname{ch}\pi u)(\operatorname{ch}\pi t + \operatorname{ch}\pi\tau)}$$

$$W_1(u,\alpha) = W_+(u,2\alpha), \quad g_1(t,\alpha) = g_+(t,2\alpha)$$

$$W_2(u,\alpha) = \frac{2\kappa\operatorname{sh}2\alpha u - 2u\sin 2\alpha}{2\kappa\operatorname{ch}2\alpha u + 2u^2 - 2u^2\cos 2\alpha + \kappa^2 + 1}, \quad \kappa = 3 - 4\nu$$

$$g_2(t,\alpha) = -g_-(t,2\alpha) + \{\sin^2\alpha(f_0[2f_1 - tf_2] - f_3[2f_2 + tf_1]) - 2(1-\nu)\sin\alpha[f_0(\sin 3\alpha - \sin\alpha\operatorname{ch}2\alpha t) - f_3\cos\alpha\operatorname{sh}2\alpha t]\}/f_4$$

$$W_\pm(u,2\alpha) = \pm\frac{\operatorname{ch}2\alpha u \mp \cos 2\alpha}{\operatorname{sh}2\alpha u \pm u\sin 2\alpha},$$

$$g_\pm(t,2\alpha) = (\operatorname{cth}\alpha t)^{\pm 1}\frac{\sin^2 2\alpha}{\operatorname{ch}2\alpha t \mp \cos 4\alpha}$$

$$f_0 = \kappa\operatorname{sh}2\alpha t\cos 2\alpha - t\sin 2\alpha,$$

$$f_1 = \cos 2\alpha + \sin^2 2\alpha - \operatorname{ch}2\alpha t$$

$$f_2 = \sin 2\alpha\operatorname{th}\alpha t(1 + \cos 2\alpha),$$

$$f_3 = (\kappa\operatorname{ch}2\alpha t - 1)\sin 2\alpha$$

$$f_4 = [f_0^2 + f_3^2](\operatorname{sh}^2\alpha t + \cos^2 2\alpha) \tag{1.6}$$

For Problem C

$$K_*(r,z,x,y) = \frac{4}{\pi^2}\int_0^\infty\int_0^\infty \operatorname{sh}\pi u\left\{K_{iu}(sx)\left[\frac{W_3(u,\alpha) - W_4(u,\alpha)}{2} - \operatorname{cth}\pi u\right] + \frac{W_3(u,\alpha)\Psi_3(u,sx) - W_4(u,\alpha)\Psi_4(u,sx)}{2\operatorname{ch}(\pi u/2)}\right\}K_{iu}(sr)\cos(z-y)sdsdu \tag{1.7}$$

The functions $\psi_m(u, sx)$ ($m = 3, 4$) also satisfy Fredholm integral equation of the second kind (1.5), in the kernels of which we must put

$$W_{3,4}(u,\alpha) = W_\pm(u,\alpha), \quad g_{3,4}(t,\alpha) = g_\pm(t,\alpha) \tag{1.8}$$

Fredholm integral equations (1.5) have been analysed in Ref. 1. We will use the collocation method with Gaussian nodes to solve them numerically.

From formulae (1.1)–(1.3) we obtain the following non-linear integral equation in the function $q(r,z)$

$$Aq^\beta(r,z) + \frac{1}{2\pi\theta}\int_\Omega K(r,z,x,y)q(x,y)d\Omega = \delta + \gamma r - f(r,z), \quad (r,z) \in \Omega \tag{1.9}$$

We must add the following two integral equilibrium conditions of the punch to Eq. (1.9)

$$\int_\Omega q(x,y)d\Omega = P, \quad \int_\Omega xq(x,y)d\Omega = PH \tag{1.10}$$

which serve to define the relation between the quantities δ, γ and P, H.

2. Method of solution

We will assume that the unknown contact area Ω is contained inside a rectangle

$$S = \{(r,z) : |r-a| \le c, |z| \le b\} \quad (b \ge c)$$

which does not reach the edge of the wedge. We will introduce the following notation

$$M = (r,z), \ N = (x,y), \ g(M) = \delta + \gamma r - f(M),$$
$$K_0(M,N) = K(r,z,x,y)$$
$$w(M) = Aq^\beta(M), \ q(M) = [w(M)/A]^{\beta_0},$$
$$\beta_0 = \beta^{-1}, \ \lambda_* = (2\pi\theta A^{1/\beta})^{-1} \tag{2.1}$$

and the non-linear operators

$$\mathcal{H}\upsilon(M) = \begin{cases} H[\upsilon(M)] = \upsilon^{\beta_0}(M), & \upsilon(M) > 0 \\ 0, & \upsilon(M) \le 0 \end{cases},$$

$$\mathfrak{Q}\upsilon(M) = \begin{cases} \upsilon(M), & \upsilon(M) > 0 \\ 0, & \upsilon(M) \le 0 \end{cases} \tag{2.2}$$

Adding the non-penetration condition to Eq. (1.9), we obtain relations, which we will write, in notation (2.1), (2.2), in the form

$$w(M) + \lambda_* \mathcal{K}\mathcal{H}w(N) = g(M) \wedge w(M) > 0, \quad M \in \Omega$$

$$\lambda_* \mathcal{K}\mathcal{H}w(N) > g(M) \wedge w(M) = 0, \quad M \in (S \backslash \overline{\Omega}) \tag{2.3}$$

where $\mathcal{K}$ is an integral operator of the form

$$\mathcal{K}\upsilon = \int\limits_S K_0(M, N)\upsilon(N)dS_N$$

System (2.3) is equivalent to the single Hammerstein integral equation

$$\upsilon + \lambda_* \mathcal{K}\mathcal{H}\upsilon = g; \quad \upsilon = \upsilon(M), \quad g = g(M) \tag{2.4}$$

We will assume that a bounded region $S_0 = \{M: g(M) > 0\}$ exists, such that $\Omega \subset \bar{S}_0 \subset S$. The solution of the problem will be sought in the Banach space $\mathcal{C}(S)$ of functions that are continuous in the rectangle S, assuming that $g(M) \in \mathcal{C}(S)$.

Theorem 1. If $\upsilon_* = \upsilon_*(M) \in \mathcal{C}(S)$ is a solution of Eq. (2.4), then $w = \mathcal{Q}\upsilon_*$, $\Omega = \{M : \upsilon_*(M) > 0\}$ is the solution of system (2.3) and $\Omega \neq \varnothing$ where $S_0 \neq \varnothing$; conversely, if $w = w(M) \in \mathcal{C}(S)$ satisfies system (2.3), then

$$\upsilon_* = g - \lambda_* \mathcal{K}\mathcal{H}w, \quad M \in S \tag{2.5}$$

is the solution of Eq. (2.4).

We will rewrite Eq. (2.4) in the form

$$\upsilon = \mathcal{U}\upsilon, \quad \mathcal{U}\upsilon = g - \lambda_* \mathcal{K}\mathcal{H}\upsilon \tag{2.6}$$

Theorem 2. Suppose the function g belongs to the open sphere $B_\rho \subset \mathcal{C}(S)$ of radius ρ with centre 0, $\|g\| < r$. Suppose T is the boundary of $\bar{B}_\rho$. Then $\mathcal{U}T \subset \bar{B}_\rho$ for fairly small values of λ_*, and Eq. (2.6) has the solution $\upsilon_* \in B_\rho$.

The proofs of Theorems 1 and 2 repeat the corresponding proofs given earlier in Ref. 9. The Shauder principle[11] is used and then the fact that $\mathcal{U}$ is a completely continuous operator.

Smallness of the parameter λ_* in Theorem 2 is achieved due to the fairly large values of the parameter A.

We will solve Eq. (2.6) by the method of successive approximations. Note that the operator $\mathcal{U}$ in the closed sphere B_ρ satisfies the Lipschitz condition with constant $q_0 = \lambda_* \|\mathcal{K}\| \beta_0 \rho^{\beta_0 - 1}$. In fact,

$$\begin{aligned} \|\mathcal{U}u - \mathcal{U}\upsilon\| &= \lambda_* \|\mathcal{K}(\mathcal{H}u - \mathcal{H}\upsilon)\| \le \lambda_* \|\mathcal{K}\| \, \|\mathcal{H}u - \mathcal{H}\upsilon\| = \\ &= \lambda_* \|\mathcal{K}\| \, \|H\mathfrak{Q}u - H\mathfrak{Q}\upsilon\| \le \lambda_* \|\mathcal{K}\| q_1 \|\mathfrak{Q}u - \mathfrak{Q}\upsilon\| \\ &\le \lambda_* \|\mathcal{K}\| q_1 \|u - \upsilon\| = q_0 \|u - \upsilon\| \end{aligned} \tag{2.7}$$

for any $u, \upsilon \in B_\rho$, since the operator H (2.2) satisfies the Lipschitz condition in the set of non-negative bounded functions $\{\chi \in \mathcal{C}(S) : 0 \le \chi \le \rho\}$ with constant $q_1 = \beta_0 \rho^{\beta_0 - 1}$.

Theorem 3. Suppose the operator $\mathcal{U}$ maps the closed sphere $\beta_\rho \subset C(S)$ of radius ρ with centre 0 into itself and assume that $q_0 = \lambda_* \|\mathcal{K}\| \beta_0 \rho^{\beta_0 - 1}$. Then, for any initial element $\upsilon_0 \in B_\rho$ the successive approximations

$$\upsilon_n = \mathcal{U}\upsilon_{n-1}, \quad n = 1, 2, \ldots \tag{2.8}$$

converge to the unique solution of Eq. (2.6)

When proving Theorem 3 one must take into account that the operator $\mathcal{U}$ is compressive, and one must also use well-known proofs (given in Ref. 12, Theorems 1.1 and 1.2, and Ref. 13, p. 605, Theorem 1).

The smaller the norm $\|g\|$ (the less the indentation of the punch when there is no inclination) the smaller the number of iterations required to obtain a solution with a specified degree of accuracy.

3. Numerical analysis

We will introduce the following dimensionless quantities for the calculations

$$r_* = \frac{r-a}{b}, \quad x_* = \frac{x-a}{b}, \quad z_* = \frac{z}{b}, \quad y_* = \frac{y}{b},$$

$$\lambda = \frac{a}{b}, \quad \delta_* = \frac{\delta}{b}, \quad \varepsilon = \frac{c}{b}$$

$$A_0 = \frac{b}{2R_1}, \quad B_0 = \frac{b}{2R_2}, \quad A_* = \frac{A(2\pi\theta)^\beta}{b},$$

$$P_* = \frac{P}{2\pi\theta b^2}, \quad q_*(r_*, z_*) = \frac{q(x, y)}{2\pi\theta} \tag{3.1}$$

etc. Here the value of A_* must be such that the parameter $\lambda_* = (A_* b)^{-1/\beta}$ is sufficiently small to satisfy the conditions of Theorem 2. The asterisks are henceforth omitted. The parameter λ represents the relative distance of the punch from the edge of the wedge. Obviously $\lambda > \varepsilon$, since the rectangle S does not reach the edge of the wedge. When $\lambda = \infty$ we obtain a contact problem for an elastic half-space with a non-linear coating.

All the calculations were carried out for $\nu = 0.3$ and $\beta = 0.4$ (this value of β is borrowed from Ref. 8, where $A \sim 1$ was also taken), $\gamma = 0$, $B_0 = 0.05$ and $\varepsilon = 0.5$. The values of α, A, δ, A_0 and λ were varied. The convergence improves as A increases (a smaller number of iterations is required) and worsens as δ increases.

In Table 1 we show values of the pressure $q(x,0) \times 10^3$ and the force $P \times 10^3$ for $\alpha = 90°$, $\delta = 0.05$, $A_0 = 0.3$ and $\lambda = 1$ and different values A of the stiffness of the coating. When A = 0 (there is no coating) the calculations were carried out using the Galanov–Newton method.[3,5] A dash denotes that this node does not belong to the contact area. The points $x = \pm 0.5$ in all cases (Table 1) lie outside the contact area. As can be seen, a coating of a specified type for each of the problems with the same indentation of the punch leads to a considerable reduction of the contact pressure and of the indenting force, and also to an increase in the dimensions of the contact area. The punch in the case of Problem C is indented the easiest of all

Table 1

A	$q(x,0) \cdot 10^3$							$P \cdot 10^3$
	x = −0.375	−0.25	−0.125	0	0.125	0.25	0.375	
Problem A								
0	–	1.12	20.7	24.5	21.1	2.58	–	8.98
0.3	0.00225	1.64	4.53	5.80	4.56	1.68	0.00378	2.12
0.5	0.0110	0.727	1.90	2.44	1.91	0.734	0.0119	0.891
Problem B								
0	–	8.82	24.3	27.2	23.7	6.61	–	12.6
0.3	0.0196	1.97	5.00	6.27	4.95	1.92	0.0140	2.40
0.5	0.0177	0.790	2.00	2.54	1.99	0.782	0.0163	0.947
Problem C								
0	–	–	19.2	23.6	20.4	1.48	–	8.15
0.3	0.000478	1.53	4.36	5.63	4.42	1.60	0.00193	2.03
0.5	0.00918	0.706	1.87	2.40	1.88	0.718	0.0106	0.873

Table 2

λ	A_0	$P \cdot 10^3$											
		$\delta \cdot 10^2 = 2$	2.5	3	3.5	2	2.5	3	3.5	2	2.5	3	3.5
		Problem A				Problem B				Problem C			
2	0.2	173	352	611	955	176	360	630	955	172	348	603	938
1	0.2	172	347	602	935	178	365	642	1019	171	341	587	909
2	0.3	145	293	511	808	147	299	526	841	145	291	506	797
1	0.3	145	291	505	795	148	302	536	858	143	287	496	776

(less force and pressure is required), and most difficult of all in the case of Problem B (rigid clamping).

In Table 2 we show values of the force $P \times 10^6$ as the function of the indentation δ for $\alpha = 90^\circ$, $A = 0.3$ and different values of δ and A_0. The force increases as the indentation increases and decreases as the ratio $A_0/B_0 = R_2/R_1$, representing the prolateness of the punch along the edge of the wedge, increases. Calculations show that in the case of Problem A the indentation of the punch is easiest of all when $\alpha = 90^\circ$ (two similar punches in the half-space), which can be explained by the extremum of the function $W_1(u,\alpha)$ (1.6) for this angle of the wedge.

For a given force, the indentation in the case of Problem B is less than in the case of Problems A and C. When A_0 increases the contact area decreases. The pressure at the point of initial contact $q(0,0)$, as a rule, increases as A_0 increases and falls as A increases. The contact area increases both when the indentation of the punch increases and when the coating parameter A increases.

For a fixed force and small values of λ in the case of Problem B, compared with the other problems, the contact pressure increases in the half of the contact area closest to the edge of the wedge (see Table 1). This shows the effect of the closeness of the edge of the wedge on the distribution of the contact pressure under the punch.

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